Algebraic stabilizations of convection-diffusion-reaction equations

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Abstract. Convection-diffusion-reaction equations appear in many mathematical models of physical, technical or biological processes and are also often used as model problems for developing numerical techniques for more complicated applications in which the interplay among convection, diffusion and reaction is important. Often, the diffusion is very small in comparison with the convection or reaction, which causes that the solutions comprise layers. It is well known that standard numerical methods then provide approximate solutions polluted by spurious oscillations unless the underlying mesh resolves the layers. Therefore, special numerical techniques have to be applied which are usually called stabilized methods. In what follows, we will consider only finite element approaches.

During the last five decades, many various stabilized methods have been developed. Their stabilizing effect can be characterized by the artificial diffusion they add to the underlying Galerkin discretization. To diminish the spurious oscillations to a sufficient extent, the artificial diffusion has to be sufficiently large. However, to avoid an excessive smearing of the layers, the artificial diffusion is not allowed to be too large. Consequently, the design of a proper stabilization is very difficult and it turns out that, due to the multiscale character of the problem, accurate approximate solutions can be obtained only if the amount of the artificial diffusion locally depends on the character of the exact solution. Consequently, numerical methods for convection-diffusion-reaction equations should be nonlinear. Many of these approaches are indeed quite successful in suppressing spurious oscillations without smearing the layers too much. However, in general, some spurious oscillations are often still present, which may be not acceptable in many applications. For example, concentrations should be in the interval $[0, 1]$ to avoid a blow-up if they are used as data of other equations. To guarantee that approximate solutions satisfy such global bounds, discretizations satisfying the discrete maximum principle (DMP) have to be used.

A recent survey revealed that there are only a few discretizations that at the same time satisfy the DMP and compute reasonably accurate solutions. An example are algebraically stabilized schemes. In contrast to many other stabilized methods which modify the variational formulation of the problem, the starting point of algebraic stabilizations is the system of linear algebraic equations corresponding to the Galerkin discretization. Then, a nonlinear algebraic term is added to the linear system in order to enforce a DMP without an excessive smearing of the layers. These methods have been intensively developed in recent years and we will formulate an abstract framework that enables the analysis of algebraically stabilized discretizations for steady-state problems in a unified way. We will present general results on local and global DMPs, existence of solutions, and error estimation. Then, various examples of algebraic stabilizations fitting into the abstract framework will be given and applied to discretizations of steady-state convection-diffusion-reaction equations. The properties of these particular algebraic stabilizations will be discussed both theoretically and by means of numerical examples.